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Use of Control Moment Gyros for the Stabilization of a Spinning Satellite

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I. Introduction

A SET of control moment gyros (CMGs) can be used as the actuator in an attitude control system by being caused to exchange angular momentum with the spacecraft. The advantages of using CMGs for the stabilization of inertially referenced satellites have become well known[1,2] during the development of the attitude control system for the Apollo Telescope Mount of NASA's Skylab Program. This paper is concerned with the feasibility of using CMGs to control a satellite which is spinning relative to a set of inertial coordinates. It is shown that the spin introduces a fundamental difference in the mechanism of opposing applied torques on the spacecraft with the CMGs of an attitude control system. This is related to the general result that the CMGs in the attitude control system of a satellite which is spinning relative to an inertial coordinate system can accumulate bias momentum only about the spin axis of the spacecraft as a result of an applied torque which is constant in spacecraft coordinates. A constant torque applied about a spacecraft axis which is perpendicular to the spin axis results in a periodic momentum requirement on the CMGs. In addition, an example is presented in which the CMG momentum requirement for a spinning satellite is less than that of a similar, nonspinning satellite.

The presentation of these concepts is organized in the following way: Sec. II includes the development of the equations which describe the form of the momentum variation of the CMGs in response to an arbitrary set of external torques; in Sec. III an example of the use of CMGs for stabilizing the attitude of an Earth-pointing satellite is presented; and Sec. IV contains some concluding remarks.

II. A Mathematical Model

The equations of motion relative to a set of coordinates which are fixed in a spacecraft containing CMGs are given by

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega} + \dot{\mathbf{H}} + \tilde{\boldsymbol{\omega}}\mathbf{H} = \mathbf{N} \tag{1}$$

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where: $\mathbf{I}$ = the 3×3 inertia tensor, $\boldsymbol{\omega}$ = a 3×1 matrix representing the angular velocity vector of the spacecraft relative to a set of inertial coordinates, $\tilde{\boldsymbol{\omega}}$ = the 3×3 skew-symmetric matrix which is isomorphic to the vector cross-product operator, $\mathbf{N}$ = a 3×1 matrix representing the total external torque vector acting on the spacecraft, and $\mathbf{H}$ = a 3×1 matrix which represents the vector sum of the individual spin angular momenta contributed by each CMG. That is, for a system containing n CMGs with the spin angular momentum of the ith CMG given by $\mathbf{h}_i$,

$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{h}_i \tag{2}$$

The dots over $\boldsymbol{\omega}$ and $\mathbf{H}$ in Eq. (1) denote the time rate of change of those vectors as measured in a set of coordinates which are fixed in the spacecraft.

Since it is desired to stabilize the spacecraft relative to a spinning reference frame let

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Delta\omega} \tag{3}$$

where $\boldsymbol{\omega}_0$ is constant. Without loss of generality it is possible to specify $\boldsymbol{\omega}_0' = [w_0\ 0\ 0]$.† That is, a spacecraft coordinate system can be selected with one axis parallel to $\boldsymbol{\omega}_0$. Of course, if this system does not coincide with the spacecraft coordinate system which is implicit in the mounting geometry of a CMG control system, then the results presented below will have to be transformed into the appropriate system in order to determine, say, gimbal excursion. Equations (3) and (1) imply

$$\mathbf{I}\boldsymbol{\Delta}\dot{\boldsymbol{\omega}} + (\tilde{\boldsymbol{\omega}}_0 + \boldsymbol{\Delta}\tilde{\boldsymbol{\omega}})\mathbf{I}(\boldsymbol{\omega}_0 + \boldsymbol{\Delta\omega}) + \dot{\mathbf{H}} + (\tilde{\boldsymbol{\omega}}_0 + \boldsymbol{\Delta}\tilde{\boldsymbol{\omega}})\mathbf{H} = \mathbf{N} \tag{4}$$

Further, if it is assumed that the CMG control system is capable of maintaining the actual state of the system identically equal to the desired state, then $\boldsymbol{\Delta\omega} = 0$ and (4) can be rewritten as

$$\dot{\mathbf{H}} + \tilde{\boldsymbol{\omega}}_0\mathbf{H} = \mathbf{N} - \boldsymbol{\omega}_0\mathbf{I}\boldsymbol{\omega}_0 = \mathbf{T} \tag{5}$$

This assumption is certainly justified for the determination of the CMG capacity which will be required to stabilize a particular attitude. The neglected terms are merely perturbations from the solution of Eq. (5).

Equation (5) is a linear, time-invariant system. Writing the Laplace transform of its expanded form yields

$$\begin{aligned} sH_x(s) &= T_x(s) + H_x(0+) \\ sH_y(s) - \omega_0 H_z(s) &= T_y(s) + H_y(0+) \\ \omega_0 H_y(s) + sH_z(s) &= T_z(s) + H_z(0+) \end{aligned} \tag{6}$$

where the initial time t_0 is taken to be $t_0 = 0+$. Notice that the spin axis momentum (H_x) is merely the integral of the applied torque. This is to be expected since this axis is fixed relative to a set of inertial coordinates. However, the solutions for $H_y(s)$ and $H_z(s)$ are given by

$$\begin{aligned} H_y(s) &= \{s[T_y(s) + H_y(0+)] + \omega_0[T_z(s) + H_z(0+)]\}/(s^2 + \omega_0^2) \\ H_z(s) &= \{-\omega_0[T_y(s) + H_y(0+)] + s[T_z(s) + H_z(0+)]\}/(s^2 + \omega_0^2) \end{aligned} \tag{7}$$

Three aspects of Eqs. (6) and (7) are particularly important in distinguishing this type of system response from the variation of $\mathbf{H}$ in an inertial attitude: 1) In the spinning coordinates the magnitude of $\mathbf{H}$ is the important quantity rather than $\dot{\mathbf{H}}$. For example, it can be seen from Eq. (6) that any constant torque $\mathbf{T}$ in the $y - z$ plane of the spacecraft can be offset by setting $H_y(t) = T_z/\omega_0$ and $H_z(t) = -T_y/\omega_0$. Thus, there is a fundamental difference in the way that a

† The prime denotes the transpose of the matrix $\boldsymbol{\omega}_0$.

CMG in a spinning satellite opposes external torques on the spacecraft. 2) The usual disturbance torques (i.e., gravity-gradient and aerodynamic torques) acting on a spinning satellite held in a prescribed attitude are either constant or periodic. Of these only a periodic applied torque of frequency ω_0 would cause a true bias momentum. 3) The y and z axis response to an impulsive torque on either axis is oscillatory.

III. An Example

The concepts expressed above are illustrated by an example. Consider a satellite which is in circular orbit and kept in alignment with the following coordinate system: x axis perpendicular to the orbital plane, and positively oriented at an acute angle relative to the ecliptic North Pole, y axis in the direction of the velocity vector, and z axis completes the orthogonal, right triad. The spacecraft is assumed to possess a nondiagonal inertia tensor and considered to be subject to a constant gravity-gradient torque. The solution to Eq. (5) for this important special case is given by:

$$\begin{aligned} H_x(t) &= H_x(0+) - 3\omega_0^2 I_{yz} t \\ H_y(t) &= T_z/\omega_0 + [H_y(0+) - T_z/\omega_0]\cos\omega_0 t + [H_z(0+) + T_y/\omega_0]\sin\omega_0 t \\ H_z(t) &= -T_y/\omega_0 + [H_z(0+) + T_y/\omega_0]\cos\omega_0 t - [H_y(0+) - T_z/\omega_0]\sin\omega_0 t \end{aligned} \tag{8}$$

where: $T_y = 4\omega_0^2 I_{xz}$, $T_z = -\omega_0^2 I_{xy}$, ω_0 = the orbital rate and I_{ij}; $i, j = x, y, z$ are the appropriate elements of the inertia tensor.

Note that: 1) the oscillatory terms in the expressions for $H_y(t)$ and $H_z(t)$ can be eliminated by a proper choice of $H_y(0+)$ and $H_z(0+)$; 2) an error in the choice of $H_y(0+)$ and $H_z(0+)$ will result only in a small oscillatory momentum requirement on the CMGs; 3) the initial time could have been chosen arbitrarily, i.e., it is unrelated to orbital position.

To illustrate the magnitude of these effects, consider a spacecraft similar to that of the first Saturn V Workshop of NASA's Skylab Program. The geometry is essentially cylindrical and for this example, the axis of symmetry, x, is perpendicular to the orbital plane and the geometric y axis is along the velocity vector. The orbital altitude is taken to be 235 naut miles. For this case‡: $\omega_0 = 1.0841 \times 10^{-3}$ (sec)$^{-1}$, $I_{yz} = -19{,}871$ slugft^2, $I_{xz} = 227{,}330$ slugft^2, and $I_{xy} = 1{,}412$ slugft^2. For these values: $T_y/\omega_0 = 986$ ftlbsec, $T_z/\omega_0 = -1.5$ ftlbsec, and the increase in bias momentum per orbit for $H_x(t)$ is 406 ftlbsec.

It is of interest to compare these results to the CMG momentum variation in a corresponding inertial mode. If the same spacecraft at the same orbital altitude is now held in the following orientation: x axis (the geometric axis of cylindrical symmetry) perpendicular to the orbital plane, and positively oriented at an acute angle relative to the ecliptic North Pole, z axis along the intersection of the orbital plane with the orbital noon meridian plane and positively directed towards the geocenter, and y axis completing the orthogonal, right triad, then

$$\begin{aligned} H_x(t) &= H_x(t_0) - 42\cos(2\omega_0 t) + 32\sin(2\omega_0 t) \text{ ftlbsec} \\ H_y(t) &= H_y(t_0) + 2{,}325\Delta t/\tau + 185\sin(2\omega_0 t) \text{ ftlbsec} \\ H_z(t) &= H_z(t_0) - 15\Delta t/\tau + 185\cos(2\omega_0 t) \text{ ftlbsec} \end{aligned} \tag{9}$$

where $\Delta t = (t - t_0)$ and τ = the orbital period = $2\pi/\omega_0$. In this case, almost all the bias momentum accumulates on the y axis and it is seen that the momentum that must be dumped per orbit is 2325 ftlbsec, almost six times larger than that for the spinning configuration.

‡ These values are illustrative only and should not be considered the actual values of the Skylab Workshop.

IV. Concluding Comments

The results presented here are independent of the number, type and mounting geometry of the CMGs. For a given CMG configuration, $\dot{\mathbf{H}}$ and $\mathbf{H}$ must be chosen to satisfy Eq. (5) without violating any constraints on the gimbal angles and rates.[2,3]

The aerodynamic torque acting on a satellite in an Earth-pointing attitude would be constant if the atmospheric density were constant. However, due to the diurnal bulge effect in the atmosphere, the aerodynamic torque term, if included on the right side of Eq. (5), would contain a sinusoidal term of frequency ω_0. For example, a conservative estimate of the bias momentum accumulation about the y and z body axes of the First Skylab Workshop due to this component of the aerodynamic torque is approximately 150 ftlbsec/axis/orbit.

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Determination of Heat-Transfer Rates from Transient Surface Temperature Measurements

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Nomenclature

k = thermal conductivity
N = number of equal time intervals
q = theoretical heat-transfer rate per unit area (heat flux)
Q = theoretical heat transferred per unit area to the semi-infinite solid
T = temperature change in the solid
T_s = stepwise change in surface temperature
α = thermal diffusivity
ρc = product of density and specific heat
ϕ = Kirchhoff variable

Subscripts

i,j = quantity at $x = i\Delta x$ and $t = j\Delta t$
n = quantity computed by a numerical method
o = properties evaluated at initial uniform temperature of solid
v = quantity obtained numerically considering all properties variable
α = quantity obtained numerically considering α independent of ϕ

A WELL-KNOWN method for determining heat-transfer rates in short flow duration facilities such as shock tubes and shock tunnels is that in which heat-transfer rates are inferred from the measurement of the surface temperature history of a solid exposed to the flow. Heat conduction in the substrate of the gage consisting of a surface-temperature sensor and the solid, which is of sufficient thickness to behave as a semi-infinite medium during the testing time, is treated by

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